Surface Integral Strategies

We have learned three basic techniques for evaluating a surface integral
\[
\int \int_S \vec{F} \cdot d\vec{S}.
\]

We can compute it directly by parameterizing the surface \(S\) and “plugging” into the
definition; we can apply the Divergence theorem to turn it into a volume integral
of the divergence of \(\vec{F}\), or we can apply Stokes’ theorem when the vector field \(\vec{F}\) is
the curl of another vector field (i.e. when its divergence is everywhere 0).

The following is a quick guide to applying the three techniques. It explains when
each of them is applicable and useful, and tries to enumerate the various “tricks”
which can be used to simplify problems.

The techniques here are completely analogous to those for line integrals. Stokes’
theorem will play the role for surface integrals that the fundamental theorem did
for line integrals: it tells us how to turn the surface integral of a vector field of a
special type (a vector field \(\vec{F}\) which is the curl of a vector field \(\vec{G}\)) into the integral
over the boundary of the surface. Recall that the fundamental theorem tells us
that the line integral of \(\vec{F} = \nabla f\) over a curve is equal to a sum (which is really the
same as an integral) of \(f\) over the endpoints of the curve. The analogy extends even
further; we can determine when a vector field is a curl by looking at its divergence
in the same way we can tell when a vector field is a gradient by looking at its
curl. Finally, the divergence theorem can be used to simplify surface integrals in
the same way that Stokes’ theorem is used to simplify line integrals. It turns the
surface integral of \(\vec{F}\) over a closed surface \(S\) into the volume integral of a derivative
of \(\vec{F}\) (the divergence) over the interior of \(S\).

**Brute Force**
Applicable: always
Useful: when \(\vec{F}\) and \(S\) are not too complicated

This should generally be the method of last resort. You should apply it only
when you’re pretty sure no simplification of the problem can be effected using a
theorem. To compute the surface integral, first find a parameterization \(\vec{r}(u, v)\)
\((a \leq u \leq b; f(u) \leq v \leq g(u))\) for the surface \(S\). Then find the normal vector to the
surface \(\vec{r}_u \times \vec{r}_v\) (check to make sure you have the orientation correct) and apply the
definition:
\[
\int \int_S \vec{F} \cdot d\vec{S} = \int_a^b \int_{f(u)}^{g(u)} \vec{F}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v) \, dv \, du
\]

When you compute a surface integral in this way ALWAYS work out the normal
vector explicitly from the formula. Don’t try to take a shortcut even when you
know the DIRECTION of the normal vector a priori (e.g. when the the surface lies
on a plane).

**Divergence Theorem; Straightforward Application**
Applicable: when \(S\) is a closed surface
Useful: when $S$ is composed of several pieces or $\nabla \cdot \vec{F}$ is simple

The divergence theorem turns the surface integral around the boundary into a volume integral over the interior $E$ of the surface:

$$\int \int_S \vec{F} \cdot d\vec{S} = \int \int_E \nabla \cdot \vec{F} \ dV$$

When $S$ is composed of more than one piece, it is often easier to parameterize the interior of $S$ than the surface itself, making this a useful way to evaluate the surface integral. Also, the divergence of a function $F$ can be simpler than the original function. Sometimes a surface integral which would be all but impossible becomes a very simple volume integral via an application of this theorem.

**Divergence Theorem; “Closing the Surface”**

Applicable: always
Useful: when the surface $S$ is not closed, but $\nabla \cdot \vec{F}$ is much simpler than $\vec{F}$

Choose a surface $S'$ which closes $S$; i.e., $S$ together with $S'$ form a closed surface. Now apply the divergence theorem to the closed surface and compute the integral over $S'$ using another technique:

$$\int \int_S \vec{F} \cdot d\vec{S} = \int \int_E \nabla \cdot \vec{F} \ dV - \int \int_{S'} \vec{F} \cdot d\vec{S}$$

**Divergence Theorem; “Change of Surface”**

Applicable: when $S$ is a surface with one or more singularities in its interior but the divergence of $F$ is zero everywhere else
Useful: when $S$ is complicated

Apply the divergence theorem to change the surface we are integrating over. Let $S'$ be some simple surface such that the div of $F$ is 0 in the region between $S$ and $S'$. Then by the divergence theorem,

$$\int \int_S \vec{F} \cdot d\vec{S} = \int \int_{S'} \vec{F} \cdot d\vec{S}$$

This is the technique we used to prove Gauss’ theorem.

**Stokes’ Theorem; Straightforward Application**

Applicable: when $\vec{F} = \nabla \times \vec{G}$
Useful: if you know $\vec{G}$ a priori or can find it easily (usually, for this type of problem you will be explicitly asked to compute the surface integral of a curl), or when the surface $S$ is closed (in which case the integral is 0)

Use Stokes’ theorem to turn the surface integral over $S$ into a line integral over its boundary, $\partial S$:

$$\int \int_S \nabla \times \vec{F} \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

Use whatever method you like to compute the line integral.
Two Applications of Stokes’ Theorem

Applicable: when \( \vec{F} = \nabla \times \vec{G} \)

Useful: when there is a surface \( S' \) with the same boundary as \( S \) such that the surface integral of \( \vec{F} \) over \( S' \) is simpler than the integral over \( S \).

When two surfaces \( S \) and \( S' \) share the same boundary \( C \), we can apply Stokes’ theorem twice to see that

\[
\int \int_S \vec{F} \cdot d\vec{S} = \int_C \vec{G} \cdot d\vec{r} = \int \int_{S'} \vec{F} \cdot d\vec{S}
\]

This trick will often lead to a simpler surface integral; the dot product of the curl of \( \vec{F} \) with the normals to \( S' \) might simplify or the surface \( S' \) might lie in the plane \( z = 0 \) where the vector field \( \vec{F} \) is simple.